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FREDHOLM OPERATORS AND THE ESSENTIAL SPECTRUM

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Abstract

Fredholm operators are defined and their basic properties are proved. The essential spectrum of an arbitrary closed operator is considered and its invariance under several kinds of perturbation is proved.

1. Introduction

We first give a complete treatment of Fredholm operators. Most of the results are found in the literature (e.g., [4, 8]), but we have been able to substantially simplify proofs in several instances. The deepest theorem we make use of is the closed graph theorem. We are able to avoid completely the concept of the opening between two subspaces.

Next we give a proof of a classical theorem of Weyl [13] on the invariance of the essential spectrum under compact perturbations and give two definitions for general closed operators in a Banach space, one of them due to Wolf [14]. We then consider more general types of perturbations. Further results may be found in [10].

Remarks on specific results and methods are given at the end.

2. Fredholm Operators

A linear operator A from a Banach space X to a Banach space Y will be called a Fredholm operator if

1. A is closed

2. the domain $D(A)$ of A is dense in X

3. $\alpha(A)$, the dimension of the null space $N(A)$ of A, is finite

4. $R(A)$, the range of A, is closed in Y

5. $\beta(A)$, the codimension of $R(A)$ in Y, is finite

The index of a Fredholm operator A is defined as

$$i(A) = \alpha(A) - \beta(A) .$$

We now make several observations. The set of Fredholm operators from X to Y will be denoted by $\Psi(X, Y)$.

O1. If we decompose X into

$$X = N(A) \oplus X' , \tag{1}$$

where X' is a closed subspace of X, then

$$\|x\| \leq \text{const.}\ \|Ax\| \tag{2}$$

holds for all $x \in D(A) \cap X'$. This merely expresses the fact that A restricted to $D(A) \cap X'$ has as inverse defined everywhere on $R(A)$, which is a Banach space. Hence this closed inverse is bounded.

O2. If A is closed, $\alpha(A) < \infty$ and (2) holds, then $R(A)$ is closed. For if $Ax_n \longrightarrow y$, $x_n \in D(A) \cap X'$, then $x_n \longrightarrow x \in X'$. Thus $x \in D(A)$ and $Ax = y$.

O3. If $A \in \Psi(X, Y)$, there is a bounded operator A' from Y to $D(A) \cap X'$ such that

(3) $$A'A = I \quad \text{on} \quad D(A) \cap X'$$

(4) $$AA' = I \quad \text{on} \quad R(A) ,$$

while A' vanishes on any prescribed complement of $R(A)$ in Y. (Here I denotes the identity operator.)

Proof. Obvious.

O4. The operator A' also satisfies

(5) $$A'A = I + F_1 \quad \text{on} \quad D(A)$$

(6) $$AA' = I + F_2 \quad \text{on} \quad Y$$

where the operators F_i are bounded and have finite dimensional ranges.

In fact $A'A - I$ vanishes on $D(A) \cap X'$ and hence maps $D(A)$ into the image of $N(A)$. A similar statement holds for $AA' - I$.

O5. If A is closed, $D(A)$ dense and there are bounded operators A_i from Y to X and compact operators K_1 on X, K_2 on Y such that

(7) $$A_1A = I + K_1 \quad \text{on} \quad D(A)$$

(8) $$AA_2 = I + K_2 \quad \text{on} \quad Y ,$$

then $A \in \Psi(X, Y)$.

Proof. $\alpha(A) \leq \alpha(I + K_1) < \infty$ and $\beta(A) \leq \beta(I + K_2) < \infty$ (F. Riesz). We must show that $R(A)$ is closed. Suppose

$\{x_n\} \subset D(A) \cap X'$, $\|x_n\| = 1$ and $Ax_n \to 0$. Since A_1 is bounded, $(I + K_1)x_n \to 0$. Since $\{x_n\}$ is bounded, there is a subsequence (also denoted by $\{x_n\}$) for which $K_1 x_n$ converges to some element z. Thus $x_n \to -z$. Since A is closed $z \in D(A)$ and $Az = 0$, i.e., $z \in N(A)$. Since X' is closed, $z \in X'$ and hence $z = 0$. But $\|z\| = \lim \|x_n\| = 1$. This shows that (2) holds. Thus R(A) is closed by 02.

We shall also need the following important lemma.

Lemma 1. _If_ $A \in \Psi(X,Y)$ _and_ $B \in \Psi(Z,X)$, _then_ $AB \in \Psi(Z,Y)$ _and_

$$i(AB) = i(A) + i(B) \quad (9)$$

Proof. Clearly, D(AB) is dense in Z(01). Set

$$X_0 = R(B) \cap N(A)$$
$$R(B) = X_0 \oplus X_1$$
$$N(A) = X_0 \oplus X_2$$
$$X = R(B) \oplus X_2 \oplus X_3$$

Then X_o, X_2, X_3 are finite dimensional and X_1 is closed. Since $D(A)$ is dense in X, we can take X_3 to be contained in $D(A)$. Set $d_i = \dim X_i$. Then

$$\alpha(AB) = \alpha(B) + d_o$$
$$\beta(AB) = \beta(A) + d_3$$

$$d_o + d_2 = \alpha(A)$$
$$d_2 + d_3 = \beta(B)$$

These show that $\alpha(AB)$ and $\beta(AB)$ are finite and (9) holds. We must also show that $R(AB)$ is closed. Now $R(AB)$ is the range of A on $D(A) \cap X_1$. If $Ax_n \longrightarrow f$ for $\{x_n\} \subset D(A) \cap X_1$, we have by (2)

$$\|x_m - x_n\| \leq \text{const.} \ \|A(x_m - x_n)\| \longrightarrow 0$$

and since X_1 is closed, $x_n \longrightarrow x \in X_1$. Since A is closed, $x \in D(A)$ and $Ax = f$. Thus $R(AB)$ is closed.

It remains to show that AB is closed. Suppose $\{z_n\} \subset D(AB)$, $z_n \longrightarrow z$, $ABz_n \longrightarrow y$. Write

$$Bz_n = x_n^{(o)} + x_n^{(1)} ,$$

where $x_n^{(o)} \in N(A)$ and $x_n^{(1)} \in X'$. Then by (2) $x_n^{(1)} \longrightarrow x^{(1)} \in X'$. If $\|x_n^{(o)}\| \leq$ const., there is a subsequence (also denoted by $\{x_n\}$) for which $x_n^{(o)} \longrightarrow x^{(o)} \in N(A)$. Thus $Bz_n \longrightarrow x^{(o)} + x^{(1)}$ and since B is closed $z \in D(B)$ and $Bz = x^{(o)} + x^{(1)}$. Since A

is closed $x^{(o)} + x^{(1)} \in D(A)$ and $A(x^{(o)} + x^{(1)}) = y$. Thus $z \in D(AB)$ and $ABz = y$. We now show that $\{x_n^{(o)}\}$ is bounded. For if $\zeta_n = \|x_n^{(o)}\| \longrightarrow \infty$, set $u_n = \zeta_n^{-1}x_n^{(o)}$. Then $\|u_n\| = 1$ and hence there is a subsequence (also denoted by $\{u_n\}$) such that $u_n \longrightarrow u \in N(A)$. Moreover

$$B(\zeta_n^{-1}z_n)-u_n = \zeta_n^{-1}(Bz_n - x_n^{(o)}) = \zeta_n^{-1}x_n^{(1)} \longrightarrow 0 .$$

Hence $B(\zeta_n^{-1}z_n) \longrightarrow u$. Since $\zeta_n^{-1}z_n \longrightarrow 0$ and B is closed, $u = 0$. This is impossible since $\|u\| = \lim \|u_n\| = 1$. This completes the proof.

Theorem 1. <u>If</u> $A \in \Psi(X,Y)$ <u>and</u> K <u>is a compact operator from</u> X <u>to</u> Y <u>then</u> $(A+K) \in \Psi(X,Y)$ <u>and</u> $i(A+K) = i(A)$.

Proof. By O4 there is a bounded operator A' such that (5) and (6) hold. Now

$$A'(A+K) = I + F_1 + A'K \quad \text{on} \quad D(A)$$

(10) $$(A+K)A' = I + F_2 + KA' \quad \text{on} \quad Y .$$

Since A' is bounded, the operators A'K and KA' are compact. We now apply O5 to obtain that $(A+K) \in \Psi(X,Y)$.

Now A' is a bounded operator from Y to D(A). If we equip D(A) with the graph norm of A, it becomes a Banach space $\tilde{X}$ and A' becomes a Fredholm operator from Y to $\tilde{X}$. Also $A \in \Psi(\tilde{X},Y)$ with $\alpha(A)$ and $\beta(A)$ the same. Hence by (6)

(11) $$i(A) + i(A') = i(I + F_1) = 0 \text{ (Riesz)} .$$

But by (10)

$$i(A + K) + i(A') = i(I + F_2 + KA') = 0 .$$

Hence $i(A + K) = i(A)$ and the proof is complete.

Theorem 2. For $A \in \Psi(X, Y)$ there is an $\varepsilon > 0$ such that for any bounded operator T from X to Y with $\|T\| < \varepsilon$ one has $(A + T) \in \Psi(X, Y)$, $i(A + T) = i(A)$ and $\alpha(A + T) \leq \alpha(A)$.

Proof. By 04 there is a bounded operator A' satisfying (5) and (6). Then

$$A'(A + T) = I + F_1 + A'T$$

(12) $$(A + T)A' = I + F_2 + TA' .$$

Take $\varepsilon = \|A'\|^{-1}$. Then $\|A'T\| < 1$ and $\|TA'\| < 1$. Thus the operators $I + A'T$ and $I + TA'$ are invertible and

$$(I + A'T)^{-1}A'(A + T) = I + (I + A'T)^{-1}F_1$$

$$(A + T)A'(I + TA')^{-1} = I + F_2(I + TA')^{-1}$$

Thus by 05, $(A + T) \in \Psi$. By (12) and Theorem 1

$$i(A + T) + i(A') = i(I + F_2 + TA') = i(I + TA') = 0 .$$

Combining this with (11) we see that $i(A+T) = i(A)$. It remains to prove $\alpha(A+T) \leq \alpha(A)$. By 03, $A'(A+T) = I + A'T$ on $D(A) \cap X'$ and hence is one-to-one on this manifold. Moreover $N(A+T) \cap X' = \{0\}$. For if y is in this set, it is in $D(A) \cap X'$ and $(A+T)y = 0$. Thus $(I + A'T)y = 0$ showing that $y = 0$. Since

$$N(A+T) \oplus X' \subseteq X = N(A) \oplus X' ,$$

we see that $\dim N(A+T) \leq \dim N(A)$ and the proof is complete.

In the sequel we shall make continual use of the following lemma.

Lemma 2. *Let $\tilde{X}$ be a Banach space dense in X and let A be a linear operator from X to Y with $D(A) = \tilde{X}$. If $A \in \Psi(\tilde{X}, Y)$, then $A \in \Psi(X, Y)$.*

Proof. Let E be the linear operator from X to $\tilde{X}$ with $D(E) = D(A)$ and $Ex = x$ for $x \in D(E)$. One checks easily that $E \in \Psi(X, \tilde{X})$. Thus $AE \in \Psi(X, Y)$ by Lemma 1. But $A = AE$ and the conclusion follows.

An operator B from X to Y is called A-compact if $D(B) \supseteq D(A)$ and for every sequence $\{x_n\} \subseteq D(A)$ such that

$$\|x_n\| + \|Ax_n\| \leq \text{const.}$$

the sequence $\{Bx_n\}$ has a convergent subsequence.

Theorem 3. *If $A \in \Psi(X, Y)$ and B is A-compact, then* $(A+B) \in \Psi(X, Y)$ and $i(A+B) = i(A)$.

Proof. If we equip D(A) with the graph norm $\|x\| + \|Ax\|$, it becomes a Banach space $\tilde{X}$ (since A is closed). If we now consider A as an operator from $\tilde{X}$ to Y, we have $A \in \Psi(\tilde{X}, Y)$. Moreover B is compact from $\tilde{X}$ to Y. Hence by Theorem 1, $(A+B) \in \Psi(\tilde{X}, Y)$ with $i(A+B) = i(A)$. We now apply Lemma 2 to obtain $(A+B) \in \Psi(X,Y)$.

Theorem 4. For each $A \in \Psi(X,Y)$ there is an $\varepsilon > 0$ such that

$$\|Bx\| \leq \varepsilon(\|x\| + \|Ax\|) , \quad x \in D(A) ,$$

holding for any operator B from X to Y with $D(B) \supseteq D(A)$ implies that $(A+B) \in \Psi(X,Y)$, $i(A+B) = i(A)$, and $\alpha(A+B) \leq \alpha(A)$.

Proof. Similar to that of Theorem 3.

For an arbitrary operator A acting in a Banach space X the Fredholm set of A, denoted by Φ_A, is the set of those complex λ for which $A - \lambda \in \Psi(X,X)$. We have by Theorem 1 and 2

Theorem 5. The set Φ_A is open and $i(A-\lambda)$ is constant on each component.

Theorem 6. If K is a compact operator in X, then $\Phi_{A+K} = \Phi_A$ and $i(A+K-\lambda) = i(A-\lambda)$ there.

In addition one also has

Theorem 7. On each component of Φ_A, the quantities $\alpha(A-\lambda)$ and $\beta(A-\lambda)$ are constant except possibly on a discrete set of points where they take on larger values.

The proof of Theorem 7 can be made to rest upon

Lemma 3. If A and B are linear operators from X to Y with $A \in \Psi(X,Y)$ and B bounded, then there is an $\varepsilon > 0$ such that $\alpha(A - \lambda B)$ is constant for $0 < |\lambda| < \varepsilon$.

Proof. Assume first that $i(A) = 0$. Let $x_1, \ldots, x_k$ be a basis for $N(A)$ and $y_1, \ldots, y_k$ be a basis for some direct complement of $R(A)$ in Y. Let $x_1', \ldots, x_k'$ be a system of bounded linear functionals on X such that

$$x_i'(x_j) = \delta_{ij} \text{ (= Kronecker delta)} .$$

One checks easily that the operator defined by

$$\tilde{A}x = Ax + \sum_{j=1}^{k} x_j'(x)y_j$$

is continuously invertible. Now x is a solution of $(A - \lambda B)x = 0$ if and only if

$$(\tilde{A} - \lambda B)x = \sum_{i=1}^{k} x_i'(x)y_i \ .$$

For some $\varepsilon_1 > 0$, $\tilde{A} - \lambda B$ is continuously invertible for $|\lambda| < \varepsilon_1$ and hence

$$x = \tilde{A}^{-1} \sum_{j=0}^{\infty} \lambda^j (B\tilde{A}^{-1})^j \sum_{i=1}^{k} x_i'(x)y_i$$

$$= \sum_{i=1}^{k} x_i'(x) \cdot \sum_{j=0}^{\infty} \lambda^j(\tilde{A}^{-1}B)^j x_i$$

$$= \sum_{i=1}^{k} x_i'(x)f_i(\lambda) \ ,$$

where $f_i(\lambda)$ are known vector valued functions in X analytic on $|\lambda| < \varepsilon_1$. Thus

$$x'_m(x) = \sum_{i=1}^{k} x'_i(x) x'_m(f_i(\lambda))$$

or

$$\sum_{i=1}^{k} [\delta_{mi} - x'_m(f_i(\lambda))] x'_i(x) = 0 .$$

Conversely, if $\xi_1, \cdots, \xi_k$ are solutions of

(13) $$\sum_{i=1}^{k} [\delta_{mi} - x'_m(f_i(\lambda))] \xi_i = 0 ,$$

then by working back we see that

$$x = \sum_{i=1}^{k} \xi_i f_i(\lambda)$$

is a solution of $(A - \lambda B)x = 0$. Thus $\alpha(A - \lambda B)$ coincides with the number of linearly independent solutions of (13). If every coefficient in (13) vanishes identically in $|\lambda| < \varepsilon_1$, there are exactly k linearly independent solutions and $\alpha(A - \lambda B) = k$ for $|\lambda| < \varepsilon_1$. Otherwise there is a minor of largest order in the determinant of (13) which does not vanish identically. Since this minor is analytic in λ, it can vanish at most at isolated points. Thus there is an $\varepsilon > 0$ such that this minor is different from 0 in $0 < |\lambda| < \varepsilon$. The number of independent solutions of (13) is constant on this set and hence

the same is true for $\alpha(A - \lambda B)$. Thus the lemma is proved for $i(A) = 0$.

Next consider the case $i(A) > 0$. Let Z be a normed $|i(A)|$-dimensional space and let $Y_1 = Y + Z$ with norm $|y + z| = |y| \oplus |z|$ when $y \in Y$, $z \in Z$. Consider A and B operators from X to Y_1. For these spaces $i(A) = 0$ and the previous case applies. But $\alpha(A - \lambda B)$ is not changed when we replace Y by Y_1. Thus the lemma is proved in this case.

If $i(A) < 0$, let Z be an $i(A)$-dimensional space and set $X_1 = X \oplus Z$. Let $\hat{A}$ and $\hat{B}$ be extensions of A and B to X_1 which vanish on Z. Then $i(\hat{A}) = 0$ and the first case applies. We then merely observe that $\alpha(\hat{A} - \lambda\hat{B}) = \alpha(A - \lambda B) + i(A)$ and the proof is complete.

Proof of Theorem 7. Let Ω be a component of Φ_A. Let λ_o be any point in Ω where $\alpha(A - \lambda)$ has its minimum value in Ω. Then by Lemma 3 there is a neighborhood about λ_o where $\alpha(A - \lambda)$ has this constant value. Let λ_1 be any other point in Ω. It suffices to prove that there is a deleted neighborhood about λ_1 in which $\alpha(A - \lambda)$ has this minimum value. Connect λ_1 to λ_o by a smooth curve in Ω. At each point of this curve there is a deleted neighborhood in which $\alpha(A - \lambda)$ is constant (Lemma 3). Since the curve is compact, there is a finite set of such neighborhoods covering it. Since in each deleted neighborhood $\alpha(A - \lambda)$ is constant and each one overlaps with at least one other, they all have the same constant value for $\alpha(A - \lambda)$, namely the minimum value. Q.E.D.

In later applications we shall need

Lemma 4. <u>If</u> $B \in \Psi(Z, X)$, A <u>is an operator from</u> X <u>to</u> Y <u>with</u> D(A) <u>dense in</u> X, <u>and</u> $AB \in \Psi(Z, Y)$, <u>then</u> $A \in \Psi(X, Y)$.

Proof. Let B' be the operator corresponding to B described in O3. One easily checks that $B' \in \Psi(X, \tilde{Z})$ where $\tilde{Z}$ is D(B) under the graph norm of B. By hypothesis $AB \in \Psi(\tilde{Z}, Y)$ and hence $ABB' \in \Psi(X, Y)$. But $ABB' = A + AF$, where F is bounded and of finite rank (O4). Hence AF is a compact operator. We now apply Theorem 1 to conclude that $A \in \Psi(X, Y)$.

Lemma 5. <u>If</u> $A \in \Psi(X, Y)$, B <u>is an operator from</u> Z <u>to</u> X <u>with</u> D(B) <u>dense in</u> Z, <u>and</u> $AB \in \Psi(Z, Y)$, <u>then</u> $B \in \Psi(Z, X)$.

Proof. Following the same procedure as in the proof of Lemma 4, we have $A'AB \in \Psi(Z, X)$, and hence in $\Psi(\tilde{Z}, X)$. Since $A'AB = B + FB$, where F is of finite rank, the operator FB is compact from $\tilde{Z}$ to X. Hence $B \in \Psi(\tilde{Z}, X)$ (Theorem 1) and we can apply Lemma 2 to conclude that $B \in \Psi(Z, X)$.

3. Weyl's Theorem and Generalizations

Let T be a self-adjoint operator in a Hilbert space H and let K be a symmetric, completely continuous operator in H. Then $T + K$ is also self-adjoint. One might ask how $\sigma(T)$ and $\sigma(T + K)$ compare. The answer was given by H. Weyl [13].

Theorem 6. <u>If</u> λ <u>is in</u> $\sigma(T)$ <u>but not in</u> $\sigma(T + K)$, <u>it must be an isolated eigenvalue of finite multiplicity.</u>

Proof. We note first that since T is self-adjoint, it

is closed, and hence $\rho(T) \subseteq \Phi_T$. Thus if $\lambda \in \sigma(T)$ but not in $\sigma(T+K)$, it must be in $\Phi_T = \Phi_{T+K}$, for otherwise it could not be in $\rho(T+K)$. Thus $T-\mu$ is a Fredholm operator for μ in a neighborhood of λ. Thus $\alpha(T-\mu)$ is constant in some deleted neighborhood of λ (Lemma 3). Since all non-real μ are in $\rho(T)$, we have $\alpha(T-\mu) = 0$ in this deleted neighborhood.

Theorem 9. _If_ $\lambda \in \sigma(T)$ _is an isolated eigenvalue of finite multiplicity, there is a symmetric, completely continuous operator_ K _in_ H _such that_ $\lambda \in \rho(T+K)$.

Proof. We first show that $\lambda \in \Phi_T$. Since $\beta(T-\lambda) = \alpha(T-\lambda) < \infty$, the only thing which must be verified is that $R(T-\lambda)$ is closed in H, i.e., that the inequality

$$\|x\| \leq C\|(T-\lambda)x\| \tag{14}$$

holds for $x \in D(T) \cap N(T-\lambda)^{\perp}$ (01). Now λ is the only point of $\sigma(T)$ in some interval $[a,b]$, where $a < \lambda < b$. If $\{E_\lambda\}$ is the spectral family for T, then the projection $E_b - E_a$ maps into $N(T-\lambda)$. But

$$\|(T-\lambda)x\|^2 = \int_{-\infty}^{\infty} (\mu-\lambda)^2 d\|E_\mu x\|^2$$

$$\geq (b-\lambda)^2 \int_b^{\infty} d\|E_\mu x\|^2 + (a-\lambda)^2 \int_{-\infty} d\|E_\mu x\|^2$$

$$= (b-\lambda)^2\|(1-E_b)x\|^2 + (a-\lambda)^2\|E_a x\|^2 .$$

If $x \in D(T) \cap N(T-\lambda)^{\perp}$, $(E_b - E_a)x = 0$ and hence (14) holds. Now let $h_1, \cdots, h_k$ be a basis for $N(T-\lambda)$ satisfying

$$(h_i, h_j) = \delta_{ij}$$

and set

$$Kx = \sum_{j=1}^{k} (x, h_j)h_j \ .$$

Then K is bounded, symmetric and of finite rank. Thus $T + K - \lambda$ is a Fredholm operator. Since $\alpha(T+K-\lambda) = \beta(T+K-\lambda) = 0$, $\lambda \in \rho(T+K)$.

We see from Weyl's theorem that the points of $\sigma(T)$ which remain invariant under any symmetric, completely continuous perturbations are precisely those points which are not isolated eigenvalues of finite multiplicity. The set of such points is called the essential spectrum of T and denoted by $\sigma_e(T)$. Thus Theorems 8 and 9 can be written as

Theorem 10. *If T is self-adjoint and K is symmetric and completely continuous, then*

$$\sigma_e(T+K) = \sigma_e(T) \ .$$

From the proofs of Theorems 8 and 9 we also have

Corollary 1. *$\sigma_e(T)$ is the complement $C\Phi_T$ of Φ_T in the complex plane.*

For many applications it is important to generalize to arbitrary closed operators in Banach space and arbitrary

compact perturbations. It is no longer true that the invariant points of the spectrum are those which are not isolated eigenvalues of finite multiplicity. Wolf [14] defines the essential spectrum by means of property expressed in Corollary 1. We denote the essential spectrum according to this definition by $\sigma_{ew}(A)$.

Definition. For any closed operator A in a Banach space X, $\sigma_{ew}(A) = C\Phi_A$.

We have immediately by Theorem 6

Theorem 11. <u>For any completely continuous operator</u> K,

$$\sigma_{ew}(A+K) = \sigma_{ew}(A) .$$

We shall also employ another definition of essential spectrum.

Definition. For a closed operator A in a Banach space X, $\sigma_{em}(A)$ is the largest subset of $\sigma(A)$ which remains invariant under arbitrary compact perturbations.

Theorem 12. $\sigma_{em}(A)$ <u>consists of all complex</u> λ <u>except those</u> $\lambda \in \Phi_A$ <u>with</u> $i(A-\lambda) = 0$.

Proof. If $\lambda \in \Phi_A$ and $i(A-\lambda) = 0$, then the example given in the proof of Lemma 3 shows that there is an operator F of finite rank for which $\lambda \in \rho(A+F)$. If $i(A-\lambda) \neq 0$, then for every compact operator K, $i(A+K-\lambda) \neq 0$ (Theorem 6) and hence $\lambda \in \sigma(A+K)$. If $\lambda \notin \Phi_A$, then $\lambda \notin \Phi_{A+K}$ and again $\lambda \in \sigma(A+K)$.

We shall now give some theorems on the invariance of $\sigma_{ew}(A)$ and $\sigma_{em}(A)$ under different types of perturbations. Throughout we shall assume that A is a closed operator with dense domain in a Banach space X and B an operator in X with $D(B) \supseteq D(A)$.

Theorem 13. <u>If</u> B <u>is</u> A-<u>compact, then</u>

$$\sigma_{ew}(A+B) = \sigma_{ew}(A) \tag{15}$$

and

$$\sigma_{em}(A+B) = \sigma_{em}(A) \tag{16}$$

Proof. We prove the theorem by showing that $\Phi_A = \Phi_{A+B}$ and that $i(A-\lambda) = i(A+B-\lambda)$ for $\lambda \in \Phi_A$. If $\lambda \in \Phi_A$, then $\lambda \in \Phi_{A+B}$ and the index relationship holds (Theorem 3). Hence $\Phi_A \subseteq \Phi_{A+B}$. If $\lambda \in \Phi_{A+B}$, it follows, in particular, that $A+B$ is closed. Since A is closed, we have

$$\|Ax\| \leq C(\|x\| + \|(A+B)x\|) , \tag{17}$$

which expresses the fact that A is a closed operator defined everywhere on the Banach space $D(A+B) = D(A)$ under the graph norm $\|x\| + \|(A+B)x\|$. Hence it is bounded on this set. From (17) it follows that B is $(A+B)$-compact. We now apply Theorem 3 to $A+B$ with perturbation $-B$. Thus $\lambda \in \Phi_A$. Hence $\Phi_{A+B} \subseteq \Phi_A$, and the theorem is proved.

Theorem 14. *If $\lambda \in \rho(A) \cap \Phi_{A+B}$ and either $(A-\lambda)^{-1}B$ or $B(A-\lambda)^{-1}$ is A-compact then (15) holds. If, in addition, $i(A+B-\lambda) = 0$, then (16) holds.*

Proof. We employ the identities

$$(18) \qquad (A+B-\mu) - (A-\mu)(A-\lambda)^{-1}(A+B-\lambda) = (\mu-\lambda)(A-\lambda)^{-1}B$$

$$(19) \qquad (A+B-\mu) - (A+B-\lambda)(A-\mu)(A-\lambda)^{-1} = (\mu-\lambda)B(A-\lambda)^{-1}.$$

Assume that $(A-\lambda)^{-1}B$ is A-compact. We equip $D(A)$ with the norm $\|x\| + \|Ax\| + \|Bx\|$. Under it, $D(A)$ becomes a Banach space $\tilde{X}$. For if $x_n \longrightarrow x$, $Ax_n \longrightarrow y$, $Bx_n \longrightarrow z$, then $x \in D(A)$ and $Ax = y$ since A is closed, while $(A+B)x = y+z$ because $A+B$ is closed. Therefore $Bx = z$. Now if $\mu \in \Phi_{A+B}$, $A+B-\mu$ is a Fredholm operator from $\tilde{X}$ to X. Moreover, $(A-\lambda)^{-1}B$ is a compact operator from $\tilde{X}$ to X. Thus

$$(20) \qquad (A-\mu)(A-\lambda)^{-1}(A+B-\lambda)$$

is a Fredholm operator from $\tilde{X}$ to X (Theorem 1). Now $(A-\lambda)^{-1}(A+B-\lambda) \in \Psi(\tilde{X},\tilde{X})$ (Lemma 1). Thus $(A-\mu) \in \Psi(\tilde{X},X)$ (Lemma 4) and hence $(A-\mu) \in \Psi(X,X)$, i.e., $\mu \in \Phi_A$. Therefore $\Phi_{A+B} \subseteq \Phi_A$. Conversely, if $\mu \in \Phi_A$, then the operator (20) is in $\Psi(\tilde{X},X)$. Hence the same is true of $A+B-\mu$, i.e., $\mu \in \Phi_{A+B}$. Thus $\Phi_A = \Phi_{A+B}$. This proves (15). If $i(A+B-\lambda) = 0$, we have by (18)

$$i(A+B-\mu) = i(A-\mu)$$

and hence (16) holds.

Theorem 15. <u>If</u> B <u>is</u> A^2-<u>compact and</u> $\lambda \in \Phi_A \cap \Phi_{A+B}$, <u>then</u> <u>(15) holds. If, in addition,</u> $i(A-\lambda) = i(A+B-\lambda)$, <u>then</u> <u>(16) holds.</u>

Proof. Consider the norm

$$|||x||| = \|x\| + \|Ax\| + \|Bx\| + \|A^2x\| + \|BAx\|$$

on $D(A^2)$. We claim that $D(A^2)$ equiped with this norm becomes a Banach space $\tilde{X}$. In fact if $x_n \to x$, $Ax_n \to y$, $Bx_n \to z$, $A^2x_n \to u$, $BAx_n \to v$, then $x \in D(A)$ since A is closed and $Ax = y$, while $y \in D(A)$ and $Ay = u$ for the same reason. Thus $x \in D(A^2)$. Since $A+B$ is closed, $(A+B)x = y+z$ and hence $Bz = z$. The same reasoning gives $(A+B)Ax = u+v$, whence $BAx = v$. Now suppose $\mu \in \Phi_A$. Then the operator $(A+B-\lambda)(A-\mu) \in \Psi(X,X)$ by Lemma 1 and hence is in $\Psi(\tilde{X},X)$. However, when we consider B as an operator from $\tilde{X}$ to X, it becomes a compact operator. Since

$$(21) \qquad (A+B-\lambda)(A-\mu) - (A+B-\mu)(A-\lambda) = (\lambda - z)B ,$$

we see that $(A+B-\mu)(A-\lambda)$ is a Fredholm operator from $\tilde{X}$ to X (Theorem 1). Thus it is in $\Psi(X,X)$ (Lemma 2). Since $(A-\lambda) \in \Psi(X,X)$ we can apply Lemma 4 to obtain that $(A+B-\mu) \in \Psi(X,X)$. Hence $\Phi_A \subseteq \Phi_{A+B}$. The same reasoning applied in the opposite direction gives $\Phi_A = \Phi_{A+B}$. Moreover for $\mu \in \Phi_A$ we have by (21), Lemma 1 and Theorem 1

$$i(A+B-\lambda) + i(A-\mu) = i(A+B-\mu) + i(A-\lambda)$$

Hence

$$i(A-\mu) = i(A+B-\mu) .$$

Thus $\sigma_{em}(A+B) = \sigma_{em}(A)$ and the proof is complete.

We see from the proof of Theorem 15 that the assumption that B is A^2-compact can be considerably weakened.

4. Remarks

R1. In the Russian literature operators satisfying properties 1, 3 -5 are called Φ-operators, with the Φ standing for Fredholm (cf. [4]). The term Fredholm operator seems to be reserved for Φ-operators having index 0. We have added assumption 2 for convenience and do not differentiate on the basis of index.

R2. Observations O4 and O5 are due to Atkinson [1] for bounded operators (cf. also [11,12]).

R3. Lemma 1 is also due to Atkinson [1] for bounded operators. For unbounded operators it is due to Gohberg [3], although there seems to be a gap in his proof (cf. A.M.S. translation of [4]). The first part of our proof follows that of [4], but our proof that R(AB) is closed is different. Our proof that AB is a closed operator was taken from Kato [8]. Other proofs may be found in [6,8].

R4. For the histories of Theorems 1 and 2 see [4]. Our proofs borrow ideas from Seeley [12]. Our proof of $\alpha(A+T) \leq \alpha(A)$ appears to be much simpler than any found in the literature.

R5. The idea for Lemma 2 and its proof came from Kato [8].

R6. Theorems 3 and 4 as well as the device employed in obtaining them from Theorems 1 and 2 are due to Nagy [9].

R7. Generalizations of Theorems 5 - 7 can be found in [8,7]. Our proof of Theorem 7 is taken directly from [4].

R8. Lemmas 4 and 5 are apparently new.

R9. The term essential spectrum originated in [5] where it was applied to self-adjoint problems for ordinary differential equations on a half-line. In this case the essential spectrum is that part of the spectrum which remains invariant under changes in the boundary conditions. Browder [2] has proposed still another definition.

R10. Theorems 12, 14, 15 are apparently new. Similar results may be found in [2,14].

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